Geometry Of The Space Of Quantum States And Phase Space Formalism

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The Content In Broad Strokes

The talk is partitioned into two blocks $\sim 8 - 10$ min each

- Overview of the project;
- An instance of a successful collaboration with external researchers.

Overview Of The Project: Geometry

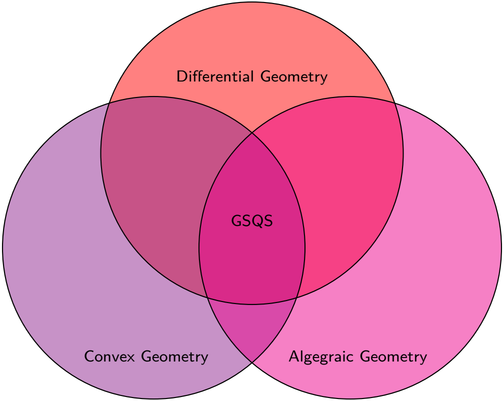
A few scenarios when the geometry of the state space is important?

- Resource quantification;
- Geometric information theory as a background for parameter estimation e.g.;
- Describing the time evolution as a geodesic motion;
- Berry (geometric) phase.

Aspects Of The Geometry

1. The space of (pure) states is a unitary (Hilbert) space by definition, thus it inherently is a TVS. In the finite dimensional scenario it is $\mathbb{C}\mathrm{P}^{\mathrm{N}-1}$;
2. The set of mixed states „*in bulk*" is a Riemannian manyfold on one hand (with appropriate distance measures);
3. On the other hand being defined by algebraic restrictions it comprises an algebraic variety;
4. The state space is a convex body embedded in some higher dimensional space.

Where is our project located



Overview Of The Project: Phase Space Formalism

The Probabilistic Nature Of Quantum Mechanics

Offers an irresistible temptation to treat the theory as a statistical theory over the phase space. However, even a shallow examination reveals that it is impossible to succeed in constructing such a theory without a push back from the quantum nature. Namely, one or the other basic properties

Zernike equation

Historically

Zernike polynomials were introduced and are still widely used in optics in the context of aberrations of optical systems.

The remarkable properties of ZP as seen from the optician perspective

- Orthogonality on a unit disk (matching the circular exit pupil of an imaging system);
- Non vanishing nature over the boundary of the disk;
- Axial $m-$symmetry;
- The most common light aberrations are described by the lowest order polynomials;
- etc.

Gut feeling:

The qualities above are not enough to explain the widespread use of these polynomials.

Characteristic equation

Consider the following diferential operator

$$\partial^2 + \alpha(\mathsf{r}\cdot\partial)^2 + \beta\,\mathsf{r}\cdot\partial, \qquad \mathsf{r} := (\mathsf{x}^1, \mathsf{x}^2), \quad \partial := (\partial_1, \partial_2).$$

It turns out that it is Hermitian

with respect to the standard invariant measure of integration over the unit disk, for the values of parmeters

$$\alpha = -1 \text{ and } \beta = -2$$

ZP's are

exactly the eigen-functions of this operator.

Moreover

Independent of the values of the parameters, its eigenvalues are real. This being a consequence of its pseudo-Hermicity.

Treating the characteristic equation as a quantum Hamiltonian

Some decade ago it has been

proposed to treat Zernike's characteristic operator as a quantum Hamiltonian

$$\widehat{\mathcal{H}} := -\hbar^2 \hat{\mathsf{Z}}^{(\alpha,\beta)} = \widehat{\mathfrak{p}}^2 + \alpha \left(\mathsf{r} \cdot \widehat{\mathfrak{p}}\right)^2 - \imath \tilde{\beta}\, \mathsf{r} \cdot \widehat{\mathfrak{p}},$$

$$\widehat{\mathcal{H}} \Psi(\mathsf{r}) = \hbar^2 \mathsf{E} \Psi(\mathsf{r}),$$

$$\widehat{\mathfrak{p}} := -\imath\hbar\partial, \quad \tilde{\beta} := \hbar\beta$$

This having far going implications, such as interpreting the Hamiltonian for the special values of parameters as that of the Higgs oscillator

Turning Zernike into a classical system

Leaving the quantum regime for a while

we will follow G.P. with coauthors who proposed a "dequantized" version of the system above. The prescription is the following

$$\mathcal{H} = \mathsf{p}^2 + \alpha\,(\mathsf{r}\cdot\mathsf{p})^2 - \imath\tilde{\beta}\,\mathsf{r}\cdot\mathsf{p}, \qquad \{\mathsf{p}_\mathsf{i}, \mathsf{x}^\mathsf{j}\} = \delta_\mathsf{i}^\mathsf{j}, \qquad \{\mathsf{p}_\mathsf{i}, \mathsf{p}_\mathsf{j}\} = \{\mathsf{x}^\mathsf{i}, \mathsf{x}^\mathsf{j}\} = 0.$$

Despite its complex appearance

Authors have succeeded in demonstrating that the system is a purely real classical (supeintegrable) system whose trajectories can be found explicitly.

We will attempt to show that this system is exactly a Higgs oscillator.

The canonical transformation

Let us rewrite the Hamiltonian in the following form

$$\mathcal{H} = \boldsymbol{\pi}^2 + \alpha\,(\mathsf{r}\cdot\boldsymbol{\pi})^2 + \frac{\tilde{\beta}^2\mathsf{r}^2}{4\,(1+\alpha\mathsf{r}^2)}, \qquad \text{where} \quad \boldsymbol{\pi} \equiv \mathsf{p} - \imath\frac{\tilde{\beta}\mathsf{r}}{2(1+\alpha\mathsf{r}^2)}.$$

In this form, the classical Zernike Hamiltonian may be interpreted as a system coupled with magnetic field defined by the vector potential

$$\partial\varphi_{\tilde{\beta}}(\mathsf{r}) = \imath\frac{\tilde{\beta}\mathsf{r}}{2(1+\alpha\mathsf{r}^2)}, \qquad \varphi_{\tilde{\beta}}(\mathsf{r}) = \imath\frac{\tilde{\beta}}{4\alpha}\log(1+\alpha\mathsf{r}^2),$$

Thus

the imaginary part of the Hamiltonian arises from this vector potential, which is a pure gauge and can be removed via an appropriate canonical transformation. As a result, we get the system with real Hamiltonian

$$(\mathsf{r}, \mathsf{p} - \partial\varphi_{\tilde{\beta}}) \to (\mathsf{r}, \mathsf{p}) : \qquad \mathcal{H} = \mathsf{p}^2 + \alpha(\mathsf{r}\cdot\mathsf{p})^2 + \frac{\tilde{\beta}^2\mathsf{r}^2}{4\,(1+\alpha\mathsf{r}^2)}.$$

Higgs oscillator

The metric

Following the standard procedure we identify the configuration space with a manyfold defined by

$$ds^2 = \sum_{i,j=1}^{2} g_{ij} dx^i dx^j = d\mathbf{r} \cdot d\mathbf{r} - \frac{\alpha(\mathbf{r} \cdot d\mathbf{r})^2}{1 + \alpha r^2}, \qquad \det g_{ij} := g = \frac{1}{1 + \alpha r^2}.$$

or embedded into three dimensional Euclidian/Minkowsky space

$$ds^2 = \sum_{i,j=1}^{2} g_{ij} dx^i dx^j = d\mathbf{r} \cdot d\mathbf{r} - \mathrm{sgn}(\alpha)(dx^0)^2, \quad r^2 - \mathrm{sgn}(\alpha)(x^0)^2 = -\frac{1}{\alpha}.$$

Higgs oscillator

The potential

appearing in the Hamiltonian is just the potential of the (pseudo)spherical Higgs oscillator with frequency $\omega = \tilde{\beta}/2$:

$$\mathsf{V}_{\tilde{\beta}} = \frac{\tilde{\beta}^2 \mathsf{r}^2}{4(1+\alpha \mathsf{r}^2)} = \frac{\tilde{\beta}^2 \mathsf{r}_0^2 \mathsf{r}^2}{4\mathsf{x}_0^2}, \quad \text{where} \quad \mathsf{r}_0^2 = \frac{1}{|\alpha|} \tag{1}$$

Thus, the de-quantized Zernike system is the Higgs oscillator and its superintegrability is not surprising.

The Hermitian momentum

It turns out,

that the momentum operator $\widehat{\mathfrak{p}}$ is non-Hermitian in the case of non-constant metrics. To overcome this issue, we define the canonical momentum operator using the following expression:

$$\widehat{\mathrm{p}} = -\imath\hbar\left(\partial + \frac{1}{2}\partial\log\sqrt{\mathrm{g}}\right) = \widehat{\mathfrak{p}} + \frac{\imath\hbar\alpha\,\mathrm{r}}{2(1+\alpha\mathrm{r}^2)},$$

Yet again,

the Hamiltonian can be expressed as:

$$\begin{aligned}\widehat{\mathcal{H}} &= \widehat{\pi}_{\mathrm{i}}\mathrm{g}^{\mathrm{ij}}\widehat{\pi}_{\mathrm{j}} + \mathrm{V}_{\tilde{\beta}-2\hbar\alpha}(\mathrm{r}) + \hbar(\tilde{\beta}-2\hbar\alpha)\\ &= \widehat{\boldsymbol{\pi}}^2 + \alpha\,(\widehat{\boldsymbol{\pi}}\cdot\mathrm{r})\,(\mathrm{r}\cdot\widehat{\boldsymbol{\pi}}) + \mathrm{V}_{\tilde{\beta}-2\hbar\alpha}(\mathrm{r}) + \hbar(\tilde{\beta}-2\hbar\alpha),\\ \widehat{\boldsymbol{\pi}} &:= \widehat{\mathrm{p}} - \partial\varphi_{\tilde{\beta}-\hbar\alpha}\end{aligned}$$

Quantum counterpart of the canonical transformation

Since the gauge

is an imaginary function, the corresponding transformation must be performed by a similarity one instead of unitary. Moreover, this transformation induces a modification of the integration measure.

$$e^{\frac{\imath}{\hbar}\varphi_{\tilde{\beta}-\hbar\alpha}}\,\hat{\boldsymbol{\pi}}\,e^{-\frac{\imath}{\hbar}\varphi_{\tilde{\beta}-\hbar\alpha}} \to \hat{\mathrm{p}}, \qquad e^{\frac{\imath}{\hbar}\varphi_{\tilde{\beta}-\hbar\alpha}}\,\Psi(\mathrm{r}) \to \widetilde{\Psi}(\mathrm{r}),$$

$$d^2\mathrm{r} \quad \to \quad e^{-\frac{2\imath}{\hbar}\varphi(\mathrm{r})}d^2\mathrm{r} = \left(1+\alpha \mathrm{r}^2\right)^{\frac{\alpha-\beta}{2\alpha}} d^2\mathrm{r}\,.$$

As a result

$$\begin{aligned}\widehat{\mathcal{H}} &= \hat{\mathrm{p}}_i g^{ij}\hat{\mathrm{p}}_j + \frac{(\tilde{\beta}-2\hbar\alpha)^2\mathrm{r}^2}{4(1+\alpha\mathrm{r}^2)} + \hbar(\tilde{\beta}-2\hbar\alpha)\\ &= \hat{\mathrm{p}}^2 + \alpha\,(\hat{\mathrm{p}}\cdot\mathrm{r})\,(\mathrm{r}\cdot\hat{\mathrm{p}}) + \frac{(\tilde{\beta}-2\hbar\alpha)^2\mathrm{r}^2}{4(1+\alpha\mathrm{r}^2)} + \hbar(\tilde{\beta}-2\hbar\alpha).\end{aligned}$$

Summerizing

- We have shown, that the imaginary part of the classical Zernike Hamiltonian may be removed by an appropriate canonical transformation induced by a purely imaginary gauge field; the resulting Hamiltonian system is just the Higgs oscillator on a (pseudo)sphere. The role of the parameters is in defining the inverse radius of the (pseudo)sphere ($\sqrt{\alpha}$) and the frequency of oscillator ($\hbar\beta/2$).
- In the quantum setup, the analogue of the gauge transformation above is a similarity transformation, i.e. "unitary" transformation with an imaginary phase. This transformation leads the initial Hamiltonian to a visibly Hermitian form. However, the resulting integration measure differs from $\sqrt{g}d^2r$. Thus the Hamiltonian is rendered as pseudo-Hermitian.
- When the Zernike parameters are $\beta = 2\alpha$ the system becomes equivalent to a free particle on a half-(pseudo)sphere with unaltered volume element and constant boundary condition on the rim, i.e., we get a Hermitian system with a Hamiltonian $\mathcal{H} = \widehat{p}_i g^{ij} \widehat{p}_j$.

Summerizing

- The conventional quantization of Higgs oscillator assumes the replacement of classical kinetic term by the Laplasian on (pseudo)sphere. In this terms the Hamiltonian takes the following form

$$\widehat{\mathcal{H}} = -\hbar^2 \Delta_{\mathrm{g}} + \frac{(\tilde{\beta} - \hbar\alpha)(\tilde{\beta} - 3\hbar\alpha)\mathrm{r}^2}{4\left(1 + \alpha \mathrm{r}^2\right)} + \hbar(\tilde{\beta} - \hbar\alpha), \qquad \text{where}$$

$$\Delta_{\mathrm{g}} = \frac{1}{\sqrt{\mathrm{g}}}\partial_{\mathrm{i}}\sqrt{\mathrm{g}}\mathrm{g}^{\mathrm{ik}}\partial_{\mathrm{j}} = \partial^2 + \alpha(\mathrm{r}\cdot\partial)^2 + \alpha(\mathrm{r}\cdot\partial).$$

While, we prefered the definition of the kinetic term as $\widehat{\mathrm{p}}_{\mathrm{i}}\mathrm{g}^{\mathrm{ij}}\widehat{\mathrm{p}}_{\mathrm{j}}$.

this work we depart from the "dequatized" version of the Hamiltonian obtained by the set of replacement rules $\widehat{\mathrm{r}} \leftrightarrow \mathrm{r}$, $\widehat{\mathrm{p}} \leftrightarrow \mathrm{p}$ together with $\widehat{\mathrm{r}}\cdot\widehat{\mathrm{p}}$ mapped to $\mathrm{r}\cdot\mathrm{p}$. On the other hand, a more widespread conventions, e.g., Wigner-Weyl approach to the mappings from the observables to functions over the phase space dictate the well-known correspondence $\mathrm{r}\cdot\mathrm{p} \leftrightarrow \frac{\widehat{\mathrm{r}}\cdot\widehat{\mathrm{p}} + \widehat{\mathrm{p}}\cdot\widehat{\mathrm{r}}}{2}$. Adoption of the later convention leads to the following Hamiltonian

$$\mathcal{H} = \mathrm{p}^2 + \alpha\,(\mathrm{r}\cdot\mathrm{p})^2 + \imath\hbar\,(2\alpha - \beta)\,\mathrm{r}\cdot\mathrm{p} + \hbar^2(\beta - \alpha). \tag{2}$$

Sumerizing

- This system has two meaningful $\hbar \to 0$ limits. The first assumes, that $\tilde{\beta} = \hbar\beta$ remains finite under the contraction, thus leading to the Hamiltonian (9). While, under the second contraction α and β are assumed finite, and the Zernike results in a free particle on a (pseudo)sphere. Thus, in both cases we deal with maximally superintegrable systems. Respectively, we can relate them with refraction index profiles provided by perfect imaging and cloaking phenomena. The second one results in the well-known Maxwell fish-eye profile, while the first one corresponds to its recently suggested modification

Thank You